Translation invariant models in QFT without ultraviolet cutoffs

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Abstract

The translation invariant model in quantum field theory is considered by functional integrations. Ultraviolet renormalization of the translation invariant Nelson model with a fixed total momentum is proven by functional integrations. As a corollary it can be shown that the Nelson Hamiltonian with zero total momentum has a ground state for arbitrary values of coupling constants in two dimension. Furthermore the ultraviolet renormalization of the polaron model is also studied.

1 Introduction

1.1 The Nelson model with a fixed total momentum

In this paper we consider an ultraviolet (UV) renormalization of the Nelson model $H(P)$ with a fixed total momentum $P \in \mathbb{R}^3$ by functional integrations.

The Nelson model describes an interaction system between a scalar bose field and particles governed by a Schrödinger operator. The interaction is linear in a field operator and the model is one of a prototype of interaction models in quantum field theory. The Nelson Hamiltonian can be realized as a self-adjoint operator H on a Hilbert space and the spectrum of H has been studied so far from several point of view. See Appendix B for the Nelson model. In the case where external potential is dropped in H, the Hamiltonian turns to be translation invariant, and it can be realized as the family of self-adjoint operators $H(P)$ indexed by the so-called total momentum $P \in \mathbb{R}^3$. The spectrum of $H(P)$ is studied for every $P \in \mathbb{R}^3$, and the difference of spectral property of $H(P)$ from every P is interesting.

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Before giving the definition of $H(P)$, we prepare tools used in this paper. The boson Fock space $\mathscr{F}$ over $L^2(\mathbb{R}^3)$ is defined by

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} [\otimes_s^n L^2(\mathbb{R}^3)]. \tag{1.1}$$

Here $\otimes_s^n L^2(\mathbb{R}^3)$ describes n fold symmetric tensor product of $L^2(\mathbb{R}^3)$ with $\otimes_s^0 L^2(\mathbb{R}^3) = \mathbb{C}$. Let $a^*(f)$ and $a(f)$, $f \in L^2(\mathbb{R}^3)$, be the creation operator and the annihilation operator, respectively, in $\mathscr{F}$, which satisfy canonical commutation relations:

$$[a(f), a^*(g)] = (\bar{f}, g), \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)].$$

Note that Here (f, g) denotes the scalar product on $L^2(\mathbb{R}^3)$ and it is linear in g and anti-linear in f. We also note that $f \mapsto a^*(f)$ and $f \mapsto a(f)$ are linear. Denote the dispersion relation by $\omega(k) = |k|$. Then the free field Hamiltonian H_f of $\mathscr{F}$ is then defined by the second quantization of ω, i.e., $H_\mathrm{f} = d\Gamma(\omega)$, and the field momentum operator P_f by $P_{\mathrm{f}\mu} = d\Gamma(k_\mu)$ and we set $P_\mathrm{f} = (P_{\mathrm{f}1}, P_{\mathrm{f}2}, P_{\mathrm{f}3})$. They satisfy

$$e^{-itH_\mathrm{f}} a^*(f) e^{-itH_\mathrm{f}} = a^*(e^{-it\omega} f), \quad e^{-itH_\mathrm{f}} a(f) e^{-itH_\mathrm{f}} = a(e^{it\omega} f). \tag{1.2}$$

The field operator $\phi = \phi(\hat{\varrho})$ on which UV cutoff $\hat{\varrho}$ is imposed is defined by

$$\phi = \frac{1}{\sqrt{2}} \left(a^*(\hat{\varrho}/\sqrt{\omega}) + a(\widetilde{\hat{\varrho}}/\sqrt{\omega}) \right). \tag{1.3}$$

Here $\hat{\varrho}$ denotes the Fourier transform of a cutoff function ϱ satisfying $\hat{\varrho}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, and $\widetilde{\hat{\varrho}}(k) = \hat{\varrho}(-k)$.

Definition 1.1 (Translation invariant Nelson Hamiltonian) $H(P)$ *is a linear operator on* $\mathscr{F}$ *and is defined by*

$$H(P) = \frac{1}{2}(P - P_\mathrm{f})^2 + H_\mathrm{f} + g\phi, \quad P \in \mathbb{R}^3. \tag{1.4}$$

Before going to discussion on $H(P)$ we have to mention the self-adjointness of $H(P)$. We decompose $H(P)$ as $H_0 + H_\mathrm{I}(P)$ to show the self-adjointness, where

$$H_0 = \frac{1}{2} {P_\mathrm{f}}^2 + H_\mathrm{f},$$
$$H_\mathrm{I}(P) = \frac{1}{2}|P|^2 - P \cdot P_\mathrm{f} + g\phi.$$

Under the assumptions

$$\hat{\varrho}/\sqrt{\omega} \in L^2(\mathbb{R}^3), \quad \hat{\varrho}/\omega \in L^2(\mathbb{R}^3), \quad \overline{\hat{\varrho}(k)} = \hat{\varrho}(-k) \tag{1.5}$$

we see that $\|\phi F\| \le (1/\sqrt{2})(2\|\hat{\varrho}/\omega\|\|H_{\rm f}^{1/2}F\| + \|\hat{\varrho}/\sqrt{\omega}\|\|F\|)$ follows for $F \in D(H_{\rm f})$ and ϕ is symmetric. Then the interaction $H_{\rm I}(P)$ is well defined, symmetric and it is infinitesimally H_0-bounded, i.e., for arbitrary $\varepsilon > 0$, there exists a $b_\varepsilon > 0$ such that

$$\|H_{\rm I}\Phi\| \le \varepsilon\|H_0\Phi\| + b_\varepsilon\|\Phi\|$$

for all $\Phi \in D(H_0)$. Thus by the Kato-Rellich theorem $H(P)$ is self-adjoint on $D(H_0)$ for every $P \in \mathbb{R}^3$. Throughout this paper we assume condition (1.5).

The purpose of this paper is to show UV renormalization (=the point charge limit) of $H(P)$. It is remarked that the point charge limit, $\hat{\varrho} \to 1\!\!1$, of $H(P)$ can be actually achieved in a similar manner to [Nel64a] by functional analysis. While the purpose of this paper is to prove the point charge limit by functional integrations. Machinery used in this paper is similar to [GHL13], where it plays an important role that e^{-tH} is positivity improving. Semi-group $e^{-tH(P)}$ is, however, not positivity improving for $P \ne 0$. Despite this fact we can achieve UV renormalization by using a diamagnetic inequality derived from functional integration.

This paper is organized as follows. In Section 2 we show UV renormalization. Section 3 is devoted to showing the existence of a renormalized ground state. In Section 4 we consider the polaron model. In Appendix we briefly introduce euclidean quantum field theory and the Nelson model.

2 Renormalization

2.1 UV renormalization and main result

Let $\lambda > 0$ be an infrared cutoff parameter and we fix it throughout. Consider the cutoff function

$$\hat{\varrho}_\varepsilon(k) = e^{-\varepsilon|k|^2/2} 1\!\!1_{|k|\ge\lambda}, \quad \varepsilon > 0 \tag{2.1}$$

and define the regularized Hamiltonian by

$$H_\varepsilon(P) = \frac{1}{2}(P - P_{\rm f})^2 + H_{\rm f} + g\phi_\varepsilon, \quad \varepsilon > 0, \tag{2.2}$$

where ϕ_ε is defined by ϕ with $\hat{\varrho}$ replaced by $\hat{\varrho}_\varepsilon$. Here $\varepsilon > 0$ is regarded as the UV cutoff parameter. We investigate the limit of $H_\varepsilon(P)$ as $\varepsilon \downarrow 0$. Precisely we can show the existence of a self-adjoint operator $H_{\rm ren}(P)$ such that

$$e^{-T(H_\varepsilon(P)-E_\varepsilon)} \to e^{-TH_{\rm ren}(P)} \tag{2.3}$$

by functional integrations, where

$$E_\varepsilon = -g^2 \int_{|k|>\lambda} \frac{e^{-\varepsilon|k|^2}}{2\omega(k)}\beta(k)dk \tag{2.4}$$

denotes the renormalization term and the propagator β is given by

$$\beta(k) = \frac{1}{\omega(k) + |k|^2/2}. \tag{2.5}$$

Notice that $E_\varepsilon \to -\infty$ as $\varepsilon \downarrow 0$. Our main theorem shows (2.3) for all $P \in \mathbb{R}^3$.

Theorem 2.1 (UV renormalization) *Let $P \in \mathbb{R}^3$. Then there exists a self-adjoint operator $H_{\text{ren}}(P)$ such that*

$$\text{s}-\lim_{\varepsilon\downarrow 0} e^{-T(H_\varepsilon(P)-E_\varepsilon)} = e^{-TH_{\text{ren}}(P)}, \quad T \geq 0. \tag{2.6}$$

We carry out the proof by functional integration and obtain E_ε as the diagonal term of a pair interaction potential on the paths of a Brownian motion.

2.2 Feynman-Kac type formula

A Feynman-Kac type formula of $(F, e^{-TH_\varepsilon(P)}G)$ is constructed for $F, G \in \mathscr{F}$ and $P \in \mathbb{R}^3$. Denote

$$H_{-k}(\mathbb{R}^n) = \{f \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^n) | \hat{f} \in L^1_{\text{loc}}(\mathbb{R}^n), \omega^{-k/2}\hat{f} \in L^2(\mathbb{R}^n)\} \tag{2.7}$$

endowed with the norm $\|f\|^2_{H_{-k}(\mathbb{R}^n)} = \displaystyle\int_{\mathbb{R}^n} |\hat{f}(x)|^2|x|^{-k}dx$. Recall that a Euclidean field is a family of Gaussian random variables $\{\phi_{\text{E}}(F), F \in H_{-1}(\mathbb{R}^4)\}$ on a probability space $(Q_{\text{E}}, \Sigma_{\text{E}}, \mu_{\text{E}})$, such that the map $F \mapsto \phi_{\text{E}}(F)$ is linear, and their mean and covariance are given by

$$\mathbb{E}_{\mu_{\text{E}}}[\phi_{\text{E}}(F)] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_{\text{E}}}[\phi_{\text{E}}(F)\phi_{\text{E}}(G)] = \frac{1}{2}(F, G)_{H_{-1}(\mathbb{R}^4)}.$$

See Appendix A for the detail. Let $(B_t)_{t\in\mathbb{R}}$ be the 3-dimensional Brownian motion on the hole real line on the Wiener space. Let $\mathbb{E}[\cdots]$ be the expectation with respect to the Wiener measure starting from zero.

Lemma 2.2 (Feynman-Kac type formula) *Let $F, G \in \mathscr{F}$. Then it follows that*

$$(F, e^{-2TH_\varepsilon(P)}G) = \mathbb{E}\left[\left(\mathrm{J}_{-T}e^{i(P-\hat{P}_{\text{f}})B_{-T}}F, e^{-\phi_{\text{E}}(\int_{-T}^T \delta_s \otimes \tilde{\varrho}_\varepsilon(\cdot - B_s)ds)}\mathrm{J}_T e^{i(P-\hat{P}_{\text{f}})B_T}G\right)_{\mathscr{E}}\right], \tag{2.8}$$

where $\hat{P}_{\text{f}} = d\Gamma(-i\nabla_k)$ and $\tilde{\varrho}_\varepsilon(x) = \left(e^{-\varepsilon|\cdot|^2/2}\mathbb{1}^\perp_\lambda/\sqrt{\omega}\right)^\vee(x)$, and $\delta_s(x) = \delta(x-s)$ is the one-dimensional Dirac delta distribution with mass on s.

Proof. See [Hir07] and Section A. □

Corollary 2.3 (Positivity improving) *Let $P = 0$. Then $e^{-TH_\varepsilon(0)}$ is positivity improving.*

Proof. By Lemma 2.2 we have

$$(F, e^{-2TH_\varepsilon(0)}G) = \mathbb{E}\left[\left(\mathrm{J}_{-T}e^{-i\hat{P}_\mathrm{f}\cdot B_{-T}}F, e^{-\phi_\mathrm{E}(\int_{-T}^{T}\delta_s\otimes\tilde{\varrho}_\varepsilon(\cdot-B_s)ds)}\mathrm{J}_T e^{-i\hat{P}_\mathrm{f}\cdot B_T}G\right)_{\mathscr{E}}\right]. \quad (2.9)$$

Since J_t and $e^{i\hat{P}_\mathrm{f}\cdot B_T}$ are positivity preserving, and $J_t^* J_s = e^{-|t-s|H_\mathrm{f}}$ is positivity improving, we have $(F, e^{-2TH_\varepsilon(0)}G) \geq 0$ for $F \geq 0$ and $G \geq 0$. We can also deduce that $(F, e^{-2TH_\varepsilon(0)}G) \neq 0$ in the same way as [Hir07]. Then $(F, e^{-2TH_\varepsilon(0)}G) > 0$ follows and it implies the statement of the lemma. □

2.3 Convergence on Fock vacuum

In order to prove Theorem 2.1 we need two ingredients:

(1) convergence (2.6) on the Fock vacuum,

(2) uniform lower bound of $H_\varepsilon(P)$ with respect to ε.

Let $\mathbb{1} = \{1, 0, 0, \cdots, \} \in \mathscr{F}$ be the Fock vacuum. In particular, for $F = \mathbb{1} = G$, we can see the corollary below.

Corollary 2.4 (Vacuum expectation) *It follows that*

$$(\mathbb{1}, e^{-2TH_\varepsilon(P)}\mathbb{1}) = \mathbb{E}\left[e^{iP\cdot(B_T - B_{-T})}e^{\frac{g^2}{2}S_\varepsilon}\right], \quad (2.10)$$

where

$$S_\varepsilon = \int_{-T}^{T} ds \int_{-T}^{T} dt W_\varepsilon(B_t - B_s, t-s) \quad (2.11)$$

is the pair interaction given by the pair potential $W_\varepsilon : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$:

$$W_\varepsilon(x,t) = \int_{|k|\geq\lambda} \frac{1}{2\omega(k)} e^{-\varepsilon|k|^2} e^{-ik\cdot x} e^{-\omega(k)|t|} dk. \quad (2.12)$$

Proof. This follows directly from Lemma 2.2. □

It can be seen that the pair potential $W_\varepsilon(B_t - B_s, t-s)$ is singular at the diagonal part $t = s$. We shall remove the diagonal part by using the Itô formula. We introduce the function

$$\varrho_\varepsilon(x,t) = \int_{|k|\geq\lambda} \frac{e^{-\varepsilon|k|^2}e^{-ik\cdot x - \omega(k)|t|}}{2\omega(k)} \beta(k) dk, \quad \varepsilon \geq 0, \quad (2.13)$$

where $\beta(k)$ is given by (2.5), and it is shown by the Itô formula that

$$\int_s^S W_\varepsilon(B_t - B_s, t-s)dt$$
$$= \varrho_\varepsilon(0,0) - \varrho_\varepsilon(B_S - B_s, S-s) + \int_s^S \nabla\varrho_\varepsilon(B_t - B_s, t-s)\cdot dB. \quad (2.14)$$

Here $\varrho_\varepsilon(0,0)$ can be regarded as the diagonal part of W_ε and turns to be a renormalization term, since $\varrho_\varepsilon(0,0) \to -\infty$ as $\varepsilon \to 0$. Let

$$S_\varepsilon^{\rm ren} = S_\varepsilon - 4T\varrho_\varepsilon(0,0), \quad \varepsilon > 0,$$

which is represented as

$$S_\varepsilon^{\rm ren} = S_\varepsilon^{OD} + 2\int_{-T}^T ds \left(\int_s^{[s+\tau]} \nabla\varrho_\varepsilon(B_t - B_s, t-s)ds\right)\cdot dB_t$$
$$- 2\int_{-T}^T \varrho_\varepsilon(B_{[s+\tau]} - B_s, [s+\tau]-s)ds. \quad (2.15)$$

Here $0 < \tau < T$ is an arbitrary number, S_ε^{OD} denotes the off-diagonal part which is given by

$$S_\varepsilon^{OD} = 2\int_{-T}^T ds \int_{[s+\tau]}^T W_\varepsilon(B_t - B_s, t-s)dt$$

and $[t] = -T \vee t \wedge T$, and the integrand is given by

$$\nabla\varrho_\varepsilon(X,t) = \int_{|k|\geq\lambda} \frac{-ike^{-ikX}e^{-|t|\omega(k)}e^{-\varepsilon|k|^2}}{2\omega(k)}\beta(k)dk.$$

Proposition 2.5 *(1) It holds that*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\left|e^{\frac{g^2}{2}S_\varepsilon^{\rm ren}} - e^{\frac{g^2}{2}S_0^{\rm ren}}\right|\right] = 0. \quad (2.16)$$

(2) There exists a constant $c > 0$ such that for all $\varepsilon \geq 0$,

$$\mathbb{E}\left[e^{\frac{g^2}{2}S_\varepsilon^{\rm ren}}\right] \leq e^{Tc} \quad (2.17)$$

Here

$$S_0^{\rm ren} = 2\int_{-T}^T ds \left(\int_{-T}^t \nabla\varrho_0(B_t - B_s, t-s)ds\right)\cdot dB_t - 2\int_{-T}^T \varrho_0(B_T - B_s, T-s)ds.$$

Proof. This can be proven by a minor modification of [GHL13, Section 2]. □

From this proposition we can derive the lemma below immediately.

Lemma 2.6 *It follows that*

$$\lim_{\varepsilon\downarrow 0}(\mathbb{1}, e^{-2T(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))}\mathbb{1}) = \mathbb{E}\left[e^{iP\cdot(B_T - B_{-T})}e^{\frac{g^2}{2}S_0^{\rm ren}}\right]. \quad (2.18)$$

2.4 Existence of ground states for $P = 0$

We shall show a uniform lower bound of $H_\varepsilon(P) + g^2\varrho_\varepsilon(0,0)$ with respect to $\varepsilon \geq 0$, and give the proof of Theorem 2.1. Let $E_\varepsilon(P) = \inf \sigma(H_\varepsilon(P))$.

Corollary 2.7 (Diamagnetic inequality) *Let $\varepsilon > 0$. Then $E_\varepsilon(0) \leq E_\varepsilon(P)$ follows for every $P \in \mathbb{R}^3$.*

PROOF. By functional integral representation (2.8) it follows that

$$|(F, e^{-TH_\varepsilon(P)}G)| \leq (|F|, e^{-TH_\varepsilon(0)}|G|).$$

This yields the inequality $E_\varepsilon(0) \leq E_\varepsilon(P)$. $\square$

Intuitively

$$\varphi_{\mathrm{g}}^T = e^{-TH_\varepsilon(0)}\mathbb{1}/\|e^{-TH_\varepsilon(0)}\mathbb{1}\|$$

is a sequence converging to a ground state. Let $\gamma(T) = (\mathbb{1}, \varphi_{\mathrm{g}}^T)^2$, i.e.,

$$\gamma(T) = \frac{(\mathbb{1}, e^{-TH_\varepsilon(0)}\mathbb{1})^2}{(\mathbb{1}, e^{-2TH_\varepsilon(0)}\mathbb{1})}. \tag{2.19}$$

The useful lemma concerning the existence and absence of ground states is the lemma below.

Lemma 2.8 *There exists a ground state of $H_\varepsilon(0)$ if and only if $\lim\limits_{T\to\infty} \gamma(T) > 0$.*

PROOF. This proof is taken from [LMS02]. Suppose that $\inf \sigma(H_\varepsilon(0)) = 0$ and set $\lim\limits_{T\to\infty} \gamma(T) = a$. Suppose that $a = 0$ and the ground state φ_{g} exists. Then $\lim_{T\to\infty} e^{-TH_\varepsilon(0)} = \mathbb{1}_{\{0\}}(H_\varepsilon(0))$. Since $\varphi_{\mathrm{g}} > 0$ by the fact that $e^{-TH_\varepsilon(0)}$ is positivity improving, it follows that $a = (\mathbb{1}, \varphi_{\mathrm{g}}) > 0$. It contradicts $a > 0$. Thus the ground state does not exist. Next suppose $a > 0$. Then $\sqrt{\gamma(T)} \geq \varepsilon$ for sufficiently large T. Let dE be the spectral measure of $H_\varepsilon(0)$. Thus we have

$$\sqrt{\gamma(T)} = \frac{\int_0^\infty e^{-Tu}dE}{(\int_0^\infty e^{-2Tu}dE)^{1/2}} \leq \frac{\int_0^\delta e^{-Tu}dE + \int_\delta^\infty e^{-Tu}dE}{(\int_0^\delta e^{-2Tu}dE)^{1/2}}.$$

Then we can derive that

$$\sqrt{\gamma(T)} \leq \frac{(\int_0^\delta e^{-2Tu}dE)^{1/2}E([0,\delta])^{1/2} + e^{-T\delta}}{(\int_0^\delta e^{-2Tu}dE)^{1/2}} = E([0,\delta])^{1/2} + \frac{1}{(\int_0^\delta e^{-2T(u-\delta)}dE)^{1/2}}.$$

Take $T \to \infty$ on both sides above, we have $\sqrt{\varepsilon} \leq E([0,\delta])^{1/2}$. Thus taking $\delta \downarrow 0$, we have $\sqrt{\varepsilon} \leq E(\{0\})^{1/2}$. Thus the ground state exists. $\square$

Using the lemma above we can show the existence of the ground state of $H_\varepsilon(0)$.

Lemma 2.9 *For all $\varepsilon > 0$, $H_\varepsilon(0)$ has the ground state and it is unique.*

PROOF. The uniqueness follows from the fact that $e^{-tH_\varepsilon(0)}$ is positivity improving. It remains to show the existence of ground state, which is proven by using Lemma 2.8. By the Feynman-Kac type formula we have

$$\gamma(T) = \frac{\left(\mathbb{E}\big[e^{\frac{g^2}{2}\int_0^T dt \int_0^T ds W_\varepsilon}\big]\right)^2}{\mathbb{E}\big[e^{\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}\big]}.$$

By the reflection symmetry of the Brownian motion we see that

$$\gamma(T) = \frac{\mathbb{E}\big[e^{\frac{g^2}{2}\int_0^T dt \int_0^T ds W_\varepsilon}\big]\mathbb{E}\big[e^{\frac{g^2}{2}\int_{-T}^0 dt \int_{-T}^0 ds W_\varepsilon}\big]}{\mathbb{E}\big[e^{\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}\big]}$$

and also the Markov property yields that

$$\gamma(T) = \frac{\mathbb{E}\big[e^{\frac{g^2}{2}\int_0^T dt \int_0^T ds W_\varepsilon + \frac{g^2}{2}\int_{-T}^0 dt \int_{-T}^0 ds W_\varepsilon}\big]}{\mathbb{E}\big[e^{\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}\big]}.$$

Then we obtain that

$$\gamma(T) = \frac{\mathbb{E}\big[e^{\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T ds W_\varepsilon - g^2\int_{-T}^0 dt \int_0^T W_\varepsilon}\big]}{\mathbb{E}\big[e^{\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}\big]}.$$

Notice that

$$\int_{-T}^0 dt \int_0^T ds W_\varepsilon \le \int_{\mathbb{R}^3} \mathbb{1}_{|k|\ge\lambda} \frac{e^{-\varepsilon|k|^2}}{\omega(k)^3}\left(1 - e^{-\omega(k)T}\right)^3 dk \le \int_{\mathbb{R}^3} \mathbb{1}_{|k|\ge\lambda} \frac{e^{-\varepsilon|k|^2}}{\omega(k)^3} dk.$$

Hence we conclude that

$$\gamma(T) \ge \exp\left(-g^2 \int_{\mathbb{R}^3} \mathbb{1}_{|k|\ge\lambda} \frac{e^{-\varepsilon|k|^2}}{\omega(k)^3} dk\right) > 0 \tag{2.20}$$

for all $T > 0$. Then the lemma follows. $\square$

2.5 Uniform lower bounds and the proof of main theorem

In this section we show a uniform lower bound of the bottom of the spectrum of $H_\varepsilon(P) + g^2\varrho_\varepsilon(0,0)$ with respect to $\varepsilon > 0$. Thanks to the diamagnetic inequality, the estimate of the uniform lower bound for any P can be reduced to that of $P = 0$. We note that the diamagnetic inequality $E(0) \le E(P)$ can be derived through a functional integration in Corollary 2.7.

Lemma 2.10 *There exists $C \in \mathbb{R}$ such that $H_\varepsilon(P) - g^2 \varrho_\varepsilon(0,0) > -C$, uniformly in $\varepsilon > 0$.*

Proof. Let $\varphi_{\rm g}$ be the ground state of $H_\varepsilon(0)$. Since $e^{-tH_\varepsilon(0)}$ is positivity improving, we see that $(\mathbb{1}, \varphi_{\rm g}) \neq 0$ and then

$$E_\varepsilon(0) - g^2 \varrho_\varepsilon(0,0) = -\lim_{T\to\infty} \frac{1}{T} \log(\mathbb{1}, e^{-T(H_\varepsilon(0) - g^2 \varrho_\varepsilon(0,0))} \mathbb{1}) > -C$$

by Proposition 2.5, where C is independent of $\varepsilon > 0$. By the diamagnetic inequality $E_\varepsilon(0) \leq E_\varepsilon(P)$ we then derive that

$$E_\varepsilon(P) - g^2 \varrho_\varepsilon(0,0) \geq -C.$$

Then the lemma follows. $\square$

Now we extend the result from Fock vacuum $\mathbb{1}$ to more general vectors of the form $F(\phi(f_1), \ldots, \phi(f_n))$, with $F \in \mathscr{S}(\mathbb{R}^n)$, where $\phi(f)$ stands for a scalar field given by

$$\phi(f) = \frac{1}{\sqrt{2}} (a^*(\hat{f}/\sqrt{\omega}) + a(\widetilde{\hat{f}}/\sqrt{\omega})).$$

Consider the subspace

$$\mathscr{D} = \left\{ F(\phi(f_1), \ldots, \phi(f_n)) | F \in \mathscr{S}(\mathbb{R}^n), f_j \in H_{-1/2}(\mathbb{R}^3), j = 1, ..., n, n \geq 1 \right\},$$

which is dense in $\mathscr{F}$.

Lemma 2.11 *(1) Let $\rho_j \in H_{-1/2}(\mathbb{R}^3)$ for $j = 1, 2$, and $\alpha, \beta \in \mathbb{C}$. Then*

$$\lim_{\varepsilon\downarrow 0} (e^{\alpha\phi(\rho_1)}, e^{-2T(H_\varepsilon(P) + g^2 \varrho_\varepsilon(0,0))} e^{\beta\phi(\rho_2)}) = \mathbb{E}\left[e^{iP\cdot(B_T - B_{-T})} e^{\frac{g^2}{2} S_0^{\rm ren} + \frac{1}{4}\xi} \right], \tag{2.21}$$

where

$$\begin{aligned}
\xi = \xi(g) &= \bar{\alpha}^2 \|\rho_1/\sqrt{\omega}\|^2 + \beta^2 \|\rho_2/\sqrt{\omega}\|^2 + 2\bar{\alpha}\beta(\rho_1/\sqrt{\omega}, e^{-2T\omega} \rho_2/\sqrt{\omega}) \\
&\quad + 2\bar{\alpha} g \int_{-T}^{T} ds \int_{\mathbb{R}^3} dk \frac{\hat{\rho}_1(k)}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\lambda} e^{-|s-T|\omega(k)} e^{-ikB_s} \\
&\quad + 2\beta g \int_{-T}^{T} ds \int_{\mathbb{R}^3} dk \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\lambda} e^{-|s+T|\omega(k)} e^{-ikB_s}.
\end{aligned}$$

(2) Let $\Phi = F(\phi(u_1), \ldots, \phi(u_n))$ and $\Psi = G(\phi(v_1), \ldots, \phi(v_m)) \in \mathscr{D}$. Then

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} (\Phi, e^{-2T(H_\varepsilon(P) + g^2 \varrho_\varepsilon(0,0))} \Psi) \\
&= (2\pi)^{-(n+m)/2} \int_{\mathbb{R}^{n+m}} dK_1 dK_2 \overline{\hat{F}(K_1)} \hat{G}(K_2) \mathbb{E}\left[e^{iP\cdot(B_T - B_{-T})} e^{\frac{g^2}{2} S_0^{\rm ren} + \frac{1}{4}\xi(K_1, K_2)} \right],
\end{aligned} \tag{2.22}$$

where

$$\xi(K_1,K_2) = -\|K_1\cdot u/\sqrt{\omega}\|^2 - \|K_2\cdot v/\sqrt{\omega}\|^2 - 2(K_1\cdot u/\sqrt{\omega}, e^{-2T\omega}K_2\cdot v/\sqrt{\omega})$$
$$- 2ig\int_{-T}^{T} ds \int_{\mathbb{R}^3} dk \frac{K_1\cdot\hat{u}(k)}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\lambda} e^{-|s-T|\omega(k)} e^{-ikB_s}$$
$$+ 2ig\int_{-T}^{T} ds \int_{\mathbb{R}^3} dk \frac{K_2\cdot\hat{v}(k)}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\lambda} e^{-|s+T|\omega(k)} e^{-ikB_s}$$

and $u = (u_1, ..., u_n)$, $v = (v_1, ..., v_m)$.

Proof. (1) follows from Lemma 2.2. (2) follows from

$$\Phi = F(\phi(u_1), \ldots, \phi(u_n)) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{F}(k_1, \cdots, k_n) e^{i\sum_{j=1}^n k_j\phi(u_j)} dk_1 \cdots dk_n$$

and Lemma 2.6. □

Now we can complete the proof of the main theorem.

Proof of Theorem 2.1. Let $F, G \in \mathscr{H}$ and $C_\varepsilon(F,G) = (F, e^{-t(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))}G)$. By Lemma 2.2 we obtain that $C_\varepsilon(F,G)$ is convergent as $\varepsilon \downarrow 0$, for every $F, G \in \mathscr{D}$. Since $\mathscr{D}$ is dense in $\mathscr{H}$, by the uniform bound $\|e^{-t(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))}\| < e^{tC}$ obtained by Lemma 2.10 we can see that $\{C_\varepsilon(F,G)\}_\varepsilon$ converges also for all $F, G \in \mathscr{H}$ by a simple approximation. Let $C_0(F,G) = \lim_{\varepsilon\downarrow 0} C_\varepsilon(F,G)$. Hence

$$|C_0(F,G)| \leq e^{tC}\|F\|\|G\|,$$

and there exists a bounded operator T_t such that

$$C_0(F,G) = (F, T_t G), \quad F, G \in \mathscr{H}$$

by the Riesz theorem. Thus $\mathrm{s}-\lim_{\varepsilon\downarrow 0} e^{-t(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))} = T_t$ follows. Furthermore, we also see that

$$\mathrm{s}-\lim_{\varepsilon\downarrow 0} e^{-t(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))} e^{-s(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))} = \mathrm{s}-\lim_{\varepsilon\downarrow 0} e^{-(t+s)(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))} = T_{t+s}.$$

Since the left-hand side above is $T_t T_s$, the semigroup property of T_t follows. Since $e^{-t(H_\varepsilon(P)+g^2\varrho_\varepsilon(0,0))}$ is a symmetric semigroup, T_t is also symmetric. Moreover by the functional integral representation (2.22) the functional $(F, T_t G)$ is continuous at $t = 0$ for every $F, G \in \mathscr{D}$. Since $\mathscr{D}$ is dense in $\mathscr{H}$ and $\|T_t\|$ is uniformly bounded, it also follows that T_t is strongly continuous at $t = 0$. Then T_t, $t \geq 0$, is strongly continuous one-parameter symmetric semigroup. Thus the semigroup version of Stone's theorem [LHB11, Proposition 3.26] implies that there exists a self-adjoint operator $H_{\mathrm{ren}}(P)$, bounded from below, such that

$$T_t = e^{-tH_{\mathrm{ren}}(P)}, \quad t \geq 0.$$

Hence the proof is completed by setting $E_\varepsilon = -g^2\varrho_\varepsilon(0,0)$. □

Let $E_{\mathrm{ren}}(P) = \inf\sigma(H_{\mathrm{ren}}(P))$.

Corollary 2.12 (Diamagnetic inequality) *It holds that* $E_{\rm ren}(0) \le E_{\rm ren}(P)$.

Proof. From inequality $|(F, e^{-T(H_\varepsilon(P)-E_\varepsilon)}G)| \le (|F|, e^{-T(H_\varepsilon(0)-E_\varepsilon)}|G|)$ it follows that $|(F, e^{-TH_{\rm ren}(P)}G)| \le (|F|, e^{-TH_{\rm ren}(0)}|G|)$. Then the corollary follows. □

3 Existence of renormalized ground state for $d = 2$

Let us suppose

$$d = 2.$$

In the case of $d = 2$ we can procedure the renormalization similar to the case of $d = 3$. The renormalization is however not needed in the case of $d = 2$, since $\varrho_\varepsilon(0,0)$ converges to the finite number $\varrho_0(0,0)$ as $\varepsilon \to 0$. One important conclusion of Theorem 2.1 is the existence of a ground state of $H_{\rm ren}(0)$ for $d = 2$.

Lemma 3.1 *It follows that*

$$\gamma(T) = \frac{(1\!\!1, e^{-TH_{\rm ren}(0)}1\!\!1)^2}{(1\!\!1, e^{-2TH_{\rm ren}(0)}1\!\!1)} > \exp\left(-g^2 \int_{\mathbb{R}^2} 1\!\!1_{|k|\ge\lambda} \frac{1}{\omega(k)^3} dk\right) > 0 \tag{3.1}$$

Proof. By (2.20) we have

$$\gamma(T) = \frac{(1\!\!1, e^{-TH_\varepsilon(0)}1\!\!1)^2}{(1\!\!1, e^{-2TH_\varepsilon(0)}1\!\!1)} \ge \exp\left(-g^2 \int_{\mathbb{R}^2} 1\!\!1_{|k|\ge\lambda} \frac{e^{-\varepsilon|k|^2}}{\omega(k)^3} dk\right) > 0.$$

Take the limit of $T \to \infty$ on both sides we can derive (3.1). □

Theorem 3.2 (Existence of the ground state) *For arbitrary values of* g, $H_{\rm ren}(0)$ *has a ground state* $\varphi_{\rm ren}$ *such that* $(1\!\!1, \varphi_{\rm ren}) \ne 0$.

Proof. By Lemma 3.1 we have

$$\lim_{T\to\infty} \frac{(1\!\!1, e^{-TH_{\rm ren}(0)}1\!\!1)^2}{(1\!\!1, e^{-2TH_{\rm ren}(0)}1\!\!1)} > \exp\left(-g^2 \int_{\mathbb{R}^2} 1\!\!1_{|k|\ge\lambda} \frac{1}{\omega(k)^3} dk\right) > 0. \tag{3.2}$$

On the other hand we see that

$$\lim_{T\to\infty} \frac{(1\!\!1, e^{-TH_{\rm ren}(0)}1\!\!1)^2}{(1\!\!1, e^{-2TH_{\rm ren}(0)}1\!\!1)} = \|P_g 1\!\!1\|^2,$$

where P_g denotes the projection to the subspace $\mathrm{Ker}(H_{\rm ren} - \inf\sigma(H_{\rm ren}))$. By (3.2) we derive that $\|P_g 1\!\!1\|^2 > 0$, which implies $H_{\rm ren}$ has a ground state $\varphi_{\rm ren}$ such that $(1\!\!1, \varphi_{\rm ren}) \ne 0$. □

4 Polaron model

We introduce the polaron model in this section. The polaron model is similar to $H_\varepsilon(P)$, and the UV renormalization can be seen in a similar manner to the Nelson model. The polaron Hamiltonian is defined by

$$H^{\rm pol}(P) = \frac{1}{2}(P - P_{\rm f})^2 + N + g\Phi, \quad P \in \mathbb{R}^3, \tag{4.1}$$

where N denotes the number operator and

$$\Phi = \frac{1}{\sqrt{2}}\left(a^*(\hat{\varrho}/\omega) + a(\tilde{\hat{\varrho}}/\omega)\right).$$

Note that the test function is $\hat{\varrho}/\omega$ which is different from the test function $\hat{\varrho}/\sqrt{\omega}$ of the Nelson Hamiltonian. We discuss UV renormalization of the polaron model. The discussion is however easier than that of the Nelson model. Let $\hat{\varrho}(k) = e^{-\varepsilon|\kappa|^2/2}$, and $H^{\rm pol}(P)$ with $\hat{\varrho}(k) = e^{-\varepsilon|\kappa|^2/2}$ is denoted by $H^{\rm pol}_\varepsilon(P)$. The vacuum expectation of $e^{-TH^{\rm pol}_\varepsilon(P)}$ is given by

$$(\mathbb{1}, e^{-TH^{\rm pol}_\varepsilon(P)}\mathbb{1}) = \mathbb{E}\left[e^{iP\cdot B_T} e^{\frac{g^2}{2}S^{\rm pol}_\varepsilon}\right], \tag{4.2}$$

where

$$S^{\rm pol}_\varepsilon = \int_0^T ds \int_0^T dt W^{\rm pol}_\varepsilon(B_t - B_s, t-s) \tag{4.3}$$

is the pair interaction for the polaron model and the pair potential is given by

$$W^{\rm pol}_\varepsilon(x,t) = \int_{|k|\geq\lambda} \frac{1}{2\omega(k)^2} e^{-\varepsilon|k|^2} e^{-ik\cdot x} e^{-|t|} dk. \tag{4.4}$$

We can see that

$$W^{\rm pol}_\varepsilon(x,t) = \frac{2\pi}{|x|}\int_{\lambda|x|}^\infty e^{-\varepsilon u}\frac{\sin u}{u} du e^{-|t|}.$$

Let

$$W^{\rm pol}_0(x,t) = \int_{|k|\geq\lambda} \frac{1}{2\omega(k)^2} e^{-ik\cdot x} e^{-|t|} dk$$

and we see that $W^{\rm pol}_\varepsilon(x,t) \to W^{\rm pol}_0(x,t)$ for each (x,t) as $\varepsilon \downarrow 0$. Then it holds that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\left|e^{\frac{g^2}{2}S^{\rm pol}_\varepsilon} - e^{\frac{g^2}{2}S^{\rm pol}_0}\right|\right] = 0. \tag{4.5}$$

From this we can prove the lemma below immediately. Note that any renormalization is not needed.

Lemma 4.1 *It follows that*

$$\lim_{\varepsilon\downarrow 0}(\mathbb{1}, e^{-TH_\varepsilon^{\rm pol}(P)}\mathbb{1}) = \mathbb{E}\left[e^{iP\cdot B_T}e^{\frac{g^2}{2}S_0^{\rm pol}}\right]. \tag{4.6}$$

Hence the theorem below is proven in the same way as the Nelson model.

Theorem 4.2 (UV renormalization) *Let $P\in\mathbb{R}^3$. Then there exists a self-adjoint operator $H_0^{\rm pol}(P)$ such that*

$$\mathrm{s-}\lim_{\varepsilon\downarrow 0} e^{-TH_\varepsilon^{\rm pol}(P)} = e^{-TH_0^{\rm pol}(P)}, \quad T\geq 0. \tag{4.7}$$

Corollary 4.3 (Removal of infrared cutoff) *It follows that*

$$\lim_{\lambda\to 0}(\mathbb{1}, e^{-TH_0^{\rm pol}(P)}\mathbb{1}) = \mathbb{E}\left[e^{iP\cdot B_T}e^{g^2\frac{\pi^2}{2}\int_0^T dt\int_0^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]. \tag{4.8}$$

Proof. It can be seen that

$$W_0^{\rm pol}(x,t) = \int_{|k|\geq\lambda}\frac{1}{2\omega(k)}e^{-ik\cdot x}e^{-|t|}dk \leq \frac{\pi^2+\delta}{|x|}e^{-|t|}$$

with some constant δ, and

$$\lim_{\lambda\to 0}W_0^{\rm pol}(x,t) = \frac{\pi^2}{|x|}e^{-|t|}$$

for each x. It can be also checked that $\mathbb{E}\left[e^{g^2\frac{\pi^2}{2}\int_0^T dt\int_0^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]$ is finite in the lemma below. Then the Lebesgue dominated convergence theorem yields the corollary. □

Lemma 4.4 $\mathbb{E}\left[e^{g^2\frac{\pi^2}{2}\int_0^T dt\int_0^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]$ *is finite.*

Proof. We separate $[0,T]\times[0,T]$ into two regions as

$$\int_0^T dt\int_0^T ds = \int_0^T dt\int_t^T ds + \int_0^T dt\int_0^t ds.$$

By the Schwarz inequality we have

$$\begin{aligned}
&\mathbb{E}\left[e^{g^2\frac{\pi^2}{2}\int_0^T dt\int_0^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]\\
&\leq \left(\mathbb{E}\left[e^{2g^2\frac{\pi^2}{2}\int_0^T dt\int_t^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]\right)^{1/2}\left(\mathbb{E}\left[e^{2g^2\frac{\pi^2}{2}\int_0^T dt\int_0^t ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]\right)^{1/2}\\
&= \left(\mathbb{E}\left[e^{2g^2\frac{\pi^2}{2}\int_0^T dt\int_t^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]\right)^{1/2}\left(\mathbb{E}\left[e^{2g^2\frac{\pi^2}{2}\int_0^T ds\int_s^T dt\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]\right)^{1/2}.
\end{aligned} \tag{4.9}$$

We estimate both sides of (4.9). By Jensen's inequality we have

$$\mathbb{E}\left[e^{g^2\pi^2\int_0^T dt\int_t^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]\le\int_0^T\frac{dt}{T}\mathbb{E}\left[e^{g^2\frac{\pi^2}{2}\int_t^T dtT\frac{e^{-|t-s|}}{|B_t-B_s|}}\right].$$

We estimate $\mathbb{E}\left[e^{g^2\pi^2\int_t^T dtT\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]$. Let $(\mathscr{F}_t)_{t\ge0}$ be the natural filtration of the Brownian motion $(B_t)_{t\ge0}$. We can see that

$$\mathbb{E}\left[e^{g^2\pi^2\int_t^T dsT\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]=\mathbb{E}\left[\mathbb{E}\left[e^{g^2\pi^2\int_t^T dsT\frac{e^{-|t-s|}}{|B_t-B_s|}}|\mathscr{F}_t\right]\right]$$
$$=\mathbb{E}\left[\mathbb{E}^{B_t}\left[e^{g^2\pi^2\int_t^T dsT\frac{e^{-|t-s|}}{|B_0-B_{s-t}|}}\right]\right]=\int_{\mathbb{R}^3}dy(2\pi t)^{-3/2}e^{-|y|^2/(2t)}\mathbb{E}^y\left[e^{g^2\pi^2\int_t^T dsT\frac{e^{-|t-s|}}{|B_0-B_{s-t}|}}\right]$$
$$=\int_{\mathbb{R}^3}dy(2\pi t)^{-3/2}e^{-|y|^2/(2t)}\mathbb{E}^y\left[e^{g^2\pi^2\int_t^T dsT\frac{e^{-|t-s|}}{|B_{s-t}-y|}}\right].$$

Since the potential $V(x)=|x|^{-1}$ is a Kato class potential, we have

$$\sup_y\mathbb{E}^y\left[e^{g^2\pi^2\int_t^T dsT\frac{e^{-|t-s|}}{|B_{s-t}-y|}}\right]\le e^{a(T-t)}$$

with some a. Hence $\mathbb{E}\left[e^{g^2\pi^2\int_0^T dt\int_t^T ds\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]<\infty$. Similarly it can be shown that $\mathbb{E}\left[e^{g^2\pi^2\int_0^T ds\int_s^T dt\frac{e^{-|t-s|}}{|B_t-B_s|}}\right]<\infty$ and hence (4.9) is finite. □

A Schrödinger representation and Euclidean field

In this section Hilbert spaces $H_{-1/2}(\mathbb{R}^3)$ and $H_{-1}(\mathbb{R}^4)$ are given by (2.7). It is well known that the boson Fock space $\mathscr{F}$ is unitarily equivalent to $L^2(Q,\mu)$, where this space consists of square integrable functions on a probability space (Q,Σ,μ). Consider the family of Gaussian random variables $\{\phi(f), f\in H_{-1/2}(\mathbb{R}^3)\}$ on (Q,Σ,μ) such that $\phi(f)$ is linear in $f\in H_{-1/2}(\mathbb{R}^3)$, and their mean and covariance are given by

$$\mathbb{E}_\mu[\phi(f)]=0\quad\text{and}\quad\mathbb{E}_\mu[\phi(f)\phi(g)]=\frac{1}{2}(f,g)_{H_{-1/2}(\mathbb{R}^3)}.$$

Given this space, the Fock vacuum $\mathbb{1}_{\mathscr{F}}$ is unitary equivalent to $\mathbb{1}_{L^2(Q)}\in L^2(Q)$, and the scalar field $\phi(f)$ is unitary equivalent to $\phi(f)$ as operators, i.e., $\phi(f)$ is regarded as multiplication by $\phi(f)$. Then the linear hull of the vectors given by the Wick products $:\prod_{j=1}^n\phi(f_j):$ is dense in $L^2(Q)$, where recall that Wick product is recursively defined by

$$:\phi(f):\;=\phi(f)$$
$$:\phi(f)\prod_{j=1}^n\phi(f_j):\;=\phi(f):\prod_{j=1}^n\phi(f_j):-\frac{1}{2}\sum_{i=1}^n(f,f_i)_{H_{-1/2}(\mathbb{R}^3)}:\prod_{j\ne i}^n\phi(f_j):$$

This allows to identify $\mathscr{F}$ and $L^2(Q)$, which we have done in (2.8), i.e., $F \in \mathscr{H}$ can be regarded as a function $\mathbb{R}^{3N} \ni x \mapsto F(x) \in L^2(Q)$ such that $\int_{\mathbb{R}^{3N}} \|F(x)\|^2_{L^2(Q)} dx < \infty$.

To construct a Feynman-Kac type formula we use a Euclidean field. Consider the family of Gaussian random variables $\{\phi_{\mathrm{E}}(F), F \in H_{-1}(\mathbb{R}^4)\}$ with mean and covariance

$$\mathbb{E}_{\mu_E}[\phi_E(F)] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_E}[\phi_E(F)\phi_E(G)] = \frac{1}{2}(F, G)_{H_{-1}(\mathbb{R}^4)}$$

on a chosen probability space (Q_E, Σ_E, μ_E). Note that for $f \in H_{-1/2}(\mathbb{R}^3)$ the relations $\delta_t \otimes f \in H_{-1}(\mathbb{R}^4)$ and $\|\delta_t \otimes f\|_{H_{-1}(\mathbb{R}^4)} = \|f\|_{H_{-1/2}(\mathbb{R}^3)}$ hold, where $\delta_t(x) = \delta(x - t)$ is Dirac delta distribution with mass on t. The family of identities used in (2.8) is then given by $\mathrm{J}_t : L^2(Q) \to \mathscr{E}$, $t \in \mathbb{R}$, defined by the relations

$$\mathrm{J}_t \mathbb{1}_{L^2(Q)} = \mathbb{1}_{\mathscr{E}}$$

and

$$\mathrm{J}_t\colon \prod_{j=1}^m \phi(f_j) : \; = \; : \prod_{j=1}^m \phi_{\mathrm{E}}(\delta_t \otimes f_j) :.$$

Under the identification $\mathscr{F} \cong L^2(Q)$ it follows that

$$(\mathrm{J}_t F, \mathrm{J}_s G)_{\mathscr{E}} = (F, e^{-|t-s|H_{\mathrm{f}}} G)_{\mathscr{F}}$$

for $F, G \in \mathscr{F}$.

B The Nelson model

The Nelson Hamiltonian H is a self-adjoint operator acting in the Hilbert space

$$L^2(\mathbb{R}^3) \otimes \mathscr{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathscr{F} \, dx,$$

which is given by

$$H = (-\frac{1}{2}\Delta + V) \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + g \int_{\mathbb{R}^3}^{\oplus} \phi(x) dx, \tag{B.1}$$

where the interaction is defined by

$$\phi(x) = \frac{1}{\sqrt{2}} \left(a^*(\hat{\varrho}/\sqrt{\omega} e^{i(\cdot, x)}) + a(\widetilde{\hat{\varrho}}/\sqrt{\omega} e^{-i(\cdot, x)}) \right).$$

H is self-adjoint on $D(H_{\mathrm{p}}) \cap D(H_{\mathrm{f}})$. A point charge limit of H, $\hat{\varrho}(k) \to \mathbb{1}$, is studied in [Nel64a, Nel64b] and recently in [GHPS12, GHL13]. It is also shown in [HHS05] that a point charge limit of H has a ground state.

We see the relationship between H and $H(P)$. The total momentum $P_{\mathrm{tot},\mu}$ is defined by $P_{\mathrm{tot},\mu} = -i\nabla_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_{\mathrm{f}\mu}$, $\mu = 1,2,3$. Let $V = 0$ in H. Then H becomes a translation invariant operator, which implies that

$$[H, P_{\mathrm{tot},\mu}] = 0, \quad \mu = 1,2,3.$$

Thus H can be decomposed with respect to the spectrum of total momentum $P_{\mathrm{tot},\mu}$ and it is known that

$$H \cong \int_{\mathbb{R}^3}^{\oplus} H(P)dP. \tag{B.2}$$

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